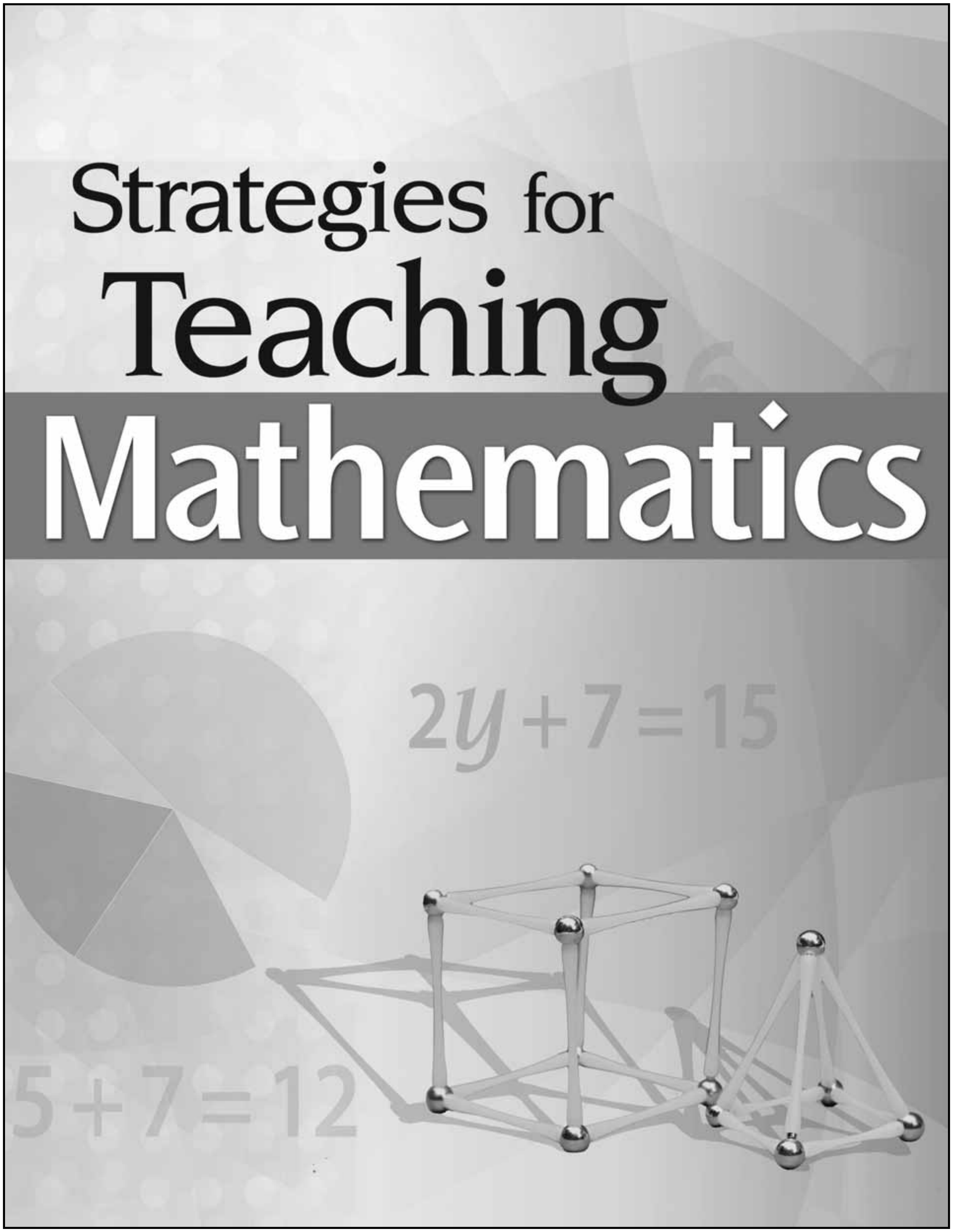

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Foreword
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SHELL EDUCATION

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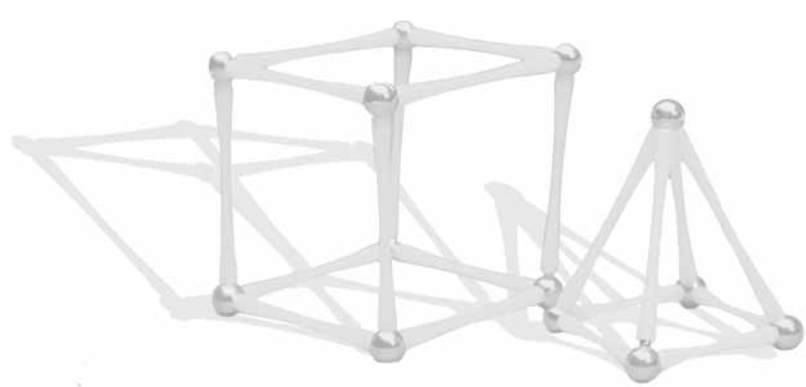
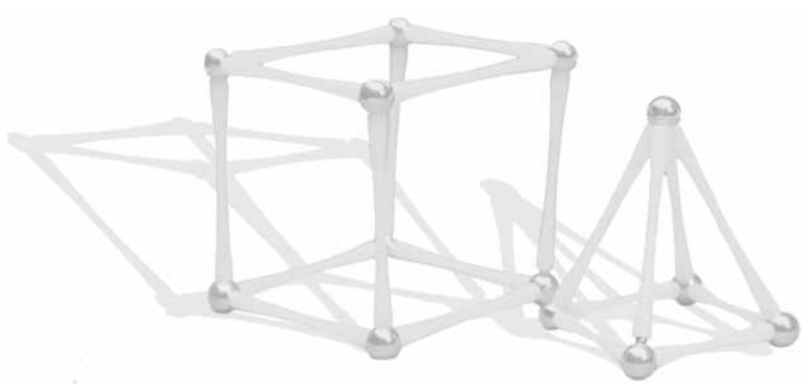


Research

Mathematics is one of the most feared subjects in school, yet it is a subject students will need for the rest of their lives. Students often struggle with learning mathematics, and teachers have long sought more effective methods for teaching it. In their landmark book, *Classroom Instruction that Works*, Robert Marzano, Debra Pickering, and Jane Pollock (2001) note that teaching has become more a science than an art. Research in the last two decades has helped educators develop a common understanding of effective instruction. This is especially evident in teaching mathematics. *Strategies for Teaching Mathematics* includes proven approaches to teaching mathematics at all levels.

Strategies for Teaching Mathematics provides educators with background information on effective instructional strategies with sample lesson plans and student reproducibles. These materials support students in truly understanding mathematical concepts, rather than just memorizing procedures. This kind of deep conceptual understanding has never been more important than it is today. Numerous educational leaders have stressed the importance of mastering 21st-century skills for today's students. Frank Levy and Richard Murnane, researchers at MIT and Harvard, found that the changing workplace has strong implications for students (2005). Computerization and globalization in today's world means that students need to be problem solvers who can think critically. No longer can students graduate from high school and expect to succeed without these qualities. Levy and Murnane note that occupations requiring higher-level problem solving have seen dramatic increases in average salary in the past 30 years, while more traditional blue-collar positions have seen even greater drops in average pay. Even more importantly, these blue-collar positions require more complex skills than similar positions of the past. Clearly, 21st-century skills need to be mastered by today's students.

So what are 21st-century skills, and what do they look like in mathematics? Students today need to understand both the procedural and conceptual foundations of mathematics. They also need to understand the "language" of mathematics. Mathematics is full of important vocabulary that has specific meanings in this context. Finally, students need to solve complex problems by making connections between prior learning and new situations. The need for students to learn these skills also means that teachers need to use new ways of teaching. Manipulatives are effective teaching tools that can be used throughout the K–12 curriculum to help students understand key mathematical concepts. New strategies for teaching essential mathematical procedures are also necessary for helping students identify real-world connections to mathematics. Finally, new methods of assessing student understanding help teachers form a stronger picture of students' skills and tailor instruction to their needs.



Research *(cont.)*

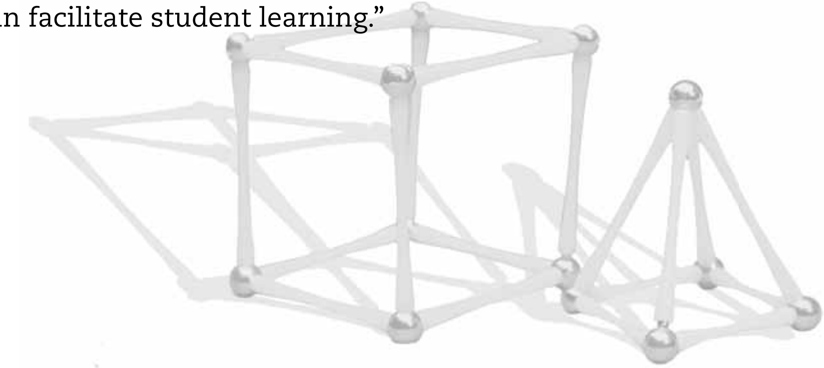
The Importance of Teaching Mathematics in a Balanced Approach

As early as 1989, educators identified the need for a balanced approach to teaching mathematics (Porter 1989). International comparisons of mathematics achievement were often made lamenting the performance of American students (Stigler and Hiebert 1997). As a result, the final decade of the 20th century was spent focusing on teaching problem-solving skills in addition to basic computation. However, new research has led to a revision of these earlier recommendations. The Mathematics Advisory Panel (U.S. Department of Education 2008), convened by the federal government, has reviewed the literature on mathematics instruction and sought the advice of key mathematics researchers. A representative study completed by Thomas Good (2008) at the University of Arizona revealed that the current mathematics curriculum in schools is too full, leading to a lack of depth of instruction and failure of students to master important concepts. Consequently, the Mathematics Advisory Panel recommended "the mutually reinforcing benefits of conceptual understanding, procedural fluency, and automatic (i.e., quick and effortless) recall of facts" (U.S. Department of Education 2008). This is different than earlier practices, because it focuses on all three areas of mathematics instruction without excluding any.

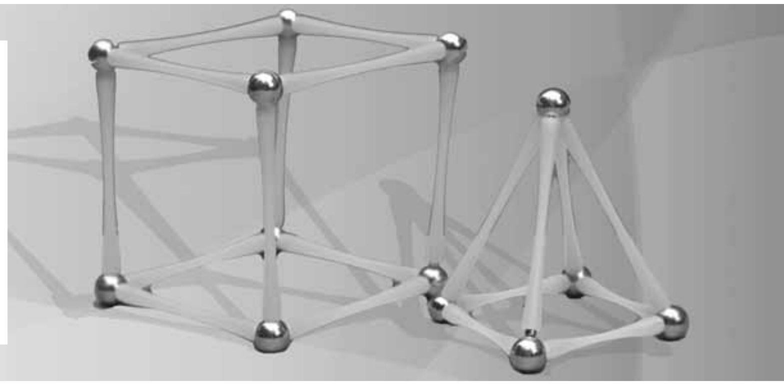
Balanced instruction includes more than just procedural and conceptual fluency. Instruction needs to be balanced to meet the needs of diverse learners as well. Phillip Schlechty (2002), in his book *Working on the Work*, suggests, "the key to school success is to be found in identifying or creating engaging schoolwork for students." Students become engaged in learning when they are taught using methods that motivate them. One way to differentiate instruction for learners is to teach with multiple intelligences in mind. Howard Gardner's (1983) groundbreaking work has helped teachers engage students who do not learn through traditional auditory methods. Strategies included in this resource such as using mathematical games and developing vocabulary meet the needs of kinesthetic and linguistic learners. Specific student activities such as Vocabulary Flip Books also engage artistic learners. Balancing mathematics instruction supports all learners in the classroom.

Differentiating Mathematics

As mathematical concepts are introduced, students learn them at different rates. Additionally, students bring different skill sets, learning styles, and prior knowledge with them to the classroom, which also cause differences in their learning. Because of these differences, a one-size-fits-all approach to mathematics instruction will not meet the needs of all students and ensure that they have conceptual understanding. However, using a differentiated approach to mathematics instruction will allow teachers to meet these needs. In their research, Strong, Thomas, Perini, and Silver (2004) note that "recognizing different mathematical learning styles and adapting differentiated teaching strategies can facilitate student learning."



Alike and Different



Standards

- understands level-appropriate vocabulary
- uses level-appropriate vocabulary in speech

Materials

- *Alike and Different* (page 35; page035.pdf)

Background Information

The Alike and Different strategy (Beck, McKeown, and Kucan 2002) gives students the opportunity to examine ways in which selected vocabulary words are both alike and different. It is best utilized after students have had some exposure to the chosen vocabulary words because students are required to analyze the key aspects of the words and make connections that deepen their understanding. This strategy can be either oral or written. Students should be provided time to discuss the words, the connections that exist among the words, and the reasons why students identified those connections.

Procedure

1. Choose a list of vocabulary words that are associated with the content lesson being taught.
2. Pair the words in a way that makes sense. You will be asking students how the pairs of words are both alike and different.
3. Write the word pairs on the board or overhead. Distribute copies of *Alike and Different* (page 35) to students and have them record each word pair on the activity sheet. (Templates for comparing four or six pairs of words are included on the Teacher Resource CD: *alike4.pdf* and *alike6.pdf*.)
4. Read the first pair of words aloud. Have students repeat the words after you.
5. Ask students to tell what they already know about each word in the first pair.
6. Ask students to think about and discuss how the word pairs are alike. You can choose to have students complete this part of the activity independently, in pairs, in small groups, or as a class. Have students record their ideas in the correct place on the chart.
7. Ask students to think about how the words are different. They should record their ideas on the chart as well.
8. Repeat steps 4–7 with the remaining pairs of words. When the activity sheet is completed, review the word pairs as a class and talk about how the words are both alike and different.

Alike and Different *(cont.)*

Example:

Word pair: *square* **and** *cube*

What students might say: A *square* and a *cube* are both shapes. They both have straight sides. A *square* is a two-dimensional shape, and a *cube* is a three-dimensional shape. A *square* has 4 vertices and a *cube* has 8 vertices.

Word pair: *radius* **and** *diameter*

What students might say: The *radius* and the *diameter* are both measurements in a circle. They are both measurements from inside of a circle. The *radius* is half of the distance of the *diameter*. The length of the *diameter* will always be larger than the *radius*.

Differentiation

Above-Level Learners

Give students the list of words and let them make pairs themselves. Then, allow them to complete the activity as directed.

Below-Level Learners

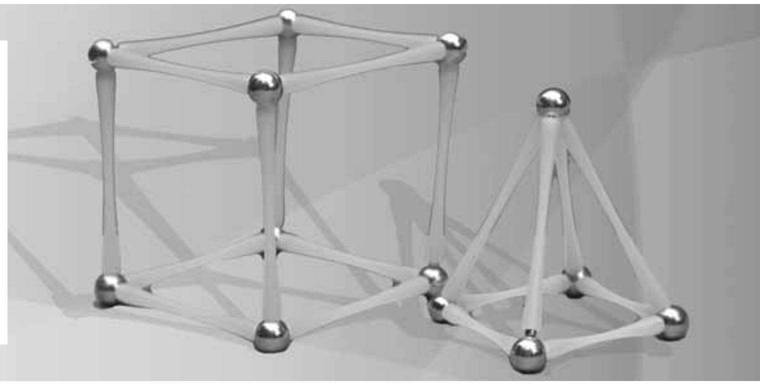
Work with students to create a more detailed list of what they know about each word by brainstorming together. Ask both broad and specific questions to help students generate ideas. Chart their responses in a way that helps students see connections (e.g., a T-chart or a Venn diagram). Then guide students in completing their charts and verbalizing ways in which the words are alike and different.

English Language Learners

Allow students to draw pictures of what they know about each of the words in the pair. Then guide students in completing their charts using both pictures and words. Allow them to use their charts to help them verbalize ways in which the words are alike and different.

Manipulatives Overview

The value of the manipulative is not in the cost, but in its use.



Common Questions About Manipulatives

What are manipulatives?

Manipulative materials are colorful, intriguing materials constructed to illustrate and model mathematical ideas and relationships and are designed to be used by students in all grades (Burns and Sibley 2000). Manipulatives are sometimes called objects to think with (Kennedy, Tipps, and Johnson 2008).

Manipulatives can take many forms in the mathematics classroom. They can be as simple as a piece of paper that is folded and cut to show congruency or as elaborate as a full-class set of base ten blocks to show place value. Fraction bars, rulers, counters, pattern blocks, algebra tiles, and tongue depressors are all examples of manipulatives. Today, there are commercial, teacher-made, student-made, and virtual manipulative materials.

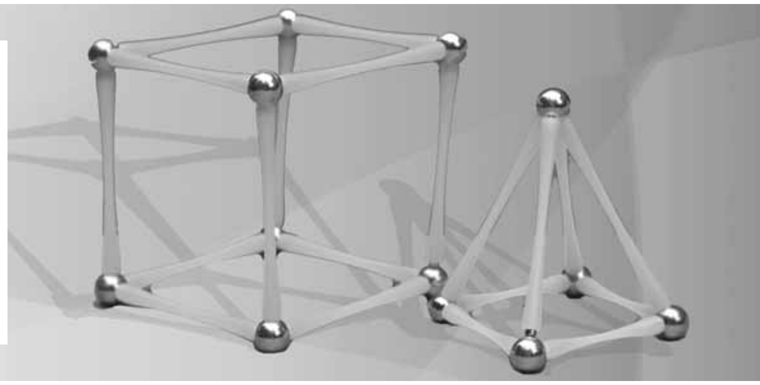
The value of the manipulative is not in the cost, but in its use. Matching the manipulative to the mathematical concept is the most important step that any teacher makes during lesson preparation.

Why should I use manipulatives?

Research indicates that lessons using manipulatives are more likely to help children achieve mathematically than lessons without manipulatives (Sowell 1989). This is because students often struggle to relate to concepts and make sense of abstract ideas without some type of first-hand experiences with them. Manipulative use gives students hands-on experiences and support learning by creating physical models that become mental models for concepts and processes (Kennedy, Tipps, and Johnson 2008).

Also, long-term use of concrete materials is positively related to increases in student mathematics achievement and improved attitudes toward mathematics (Grouws and Cebulla 2000). When students can directly relate to a concept through the use of manipulatives and feel successful, they do not fear failure and have greater success seeing relationships and connections among the five designated areas of mathematics. Using manipulatives also “helps students understand mathematical concepts and processes, increases thinking flexibility, provides tools for problem-solving, and can reduce math anxiety for some students” (The Education Alliance 2006).

Using Algebra Tiles for Collecting Like Terms



Standard

- Understands basic operations (e.g., combining like terms) on algebraic expressions

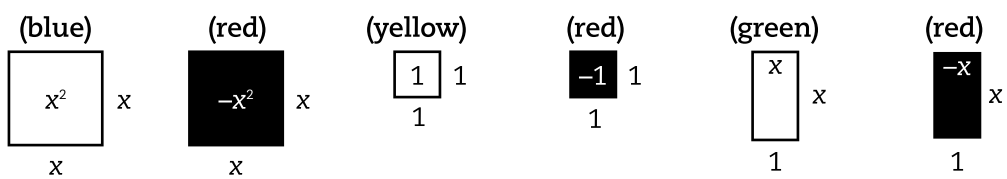
Secondary Sample Lesson

Materials

- overhead algebra tiles
- bag of algebra tiles for each student
- paper and pencils
- 1 red, yellow, green, and blue marker per pair of students
- *Combining Like Terms* (page 110; page110.pdf)

Procedure

1. Divide the class into pairs. Distribute the paper and markers to each pair.
2. Draw the diagram below on the board or overhead and have students create it on their papers.



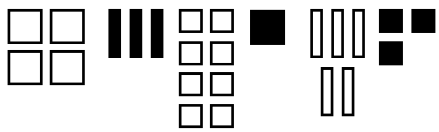
3. Distribute the bags of algebra tiles to the pairs. Have students find a tile to match each part of the diagram on their papers. As a class, discuss the tiles and their values.
4. Have a "pop quiz" by holding up each tile and asking, "What does this stand for?" Complete this several times until students are comfortable with the values of the manipulatives.
5. Explain how the tiles can be combined to make zero pairs. For example, a green rod (x) and a red rod ($-x$) combine to make a zero pair.
6. Model measuring the length and width of the blue square. They are the same. Tell the students that instead of using the actual length, they will let x represent the measurement. Tell students that the blue square is named for its area: $x \bullet x = x^2$. This area concept is the same for the other tiles.

Using Algebra Tiles for Collecting Like Terms *(cont.)*

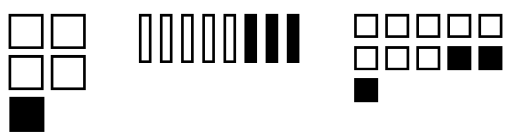
Procedure *(cont.)*

7. Write the following expression on the overhead: $4x^2 - 3x + 8 - x^2 + 5x - 3$. Have students record the same expression on their papers.
8. Model how to place the appropriate tiles under the first three terms of the expression and have students place the same tiles on their papers.
9. Have student volunteers come to the overhead projector and place algebra tiles under the remaining three terms.

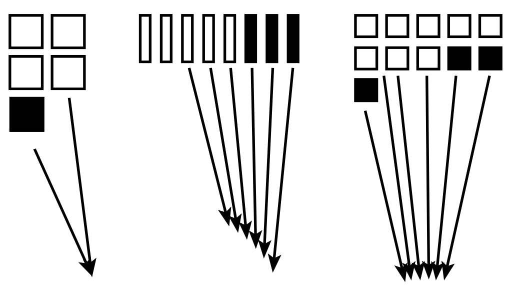
$$4x^2 - 3x + 8 - x^2 + 5x - 3$$



10. Show students how to arrange the tiles in descending order, and have them complete the same step after you model.

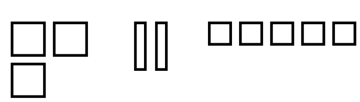


11. Then pull zero pairs to the side and have students do the same.



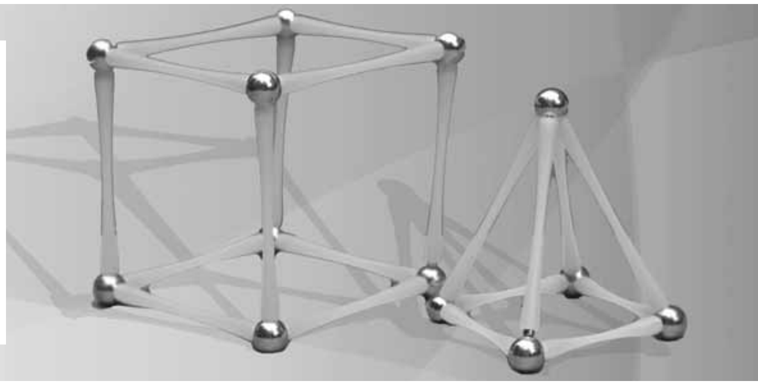
12. Look at the values of the remaining algebra tiles to determine what expression remains. $(3x^2 + 2x + 5)$

$3x^2$ $2x$ 5



Teaching Procedures Overview

In order for students to make meaning in mathematics, they must understand both the how and why.

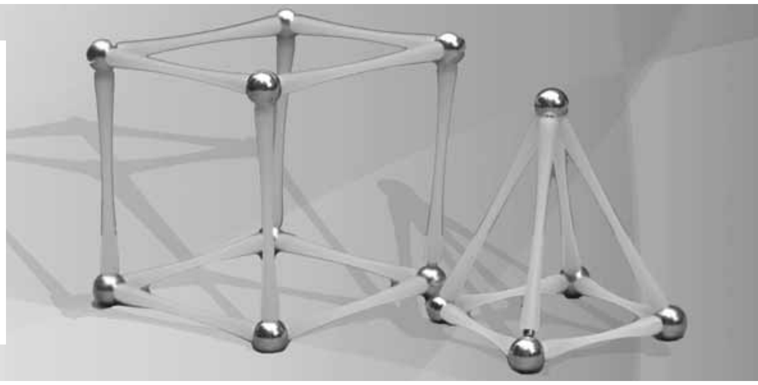


When adults think back to their experiences in mathematics instruction, they often remember the procedures being drilled into their heads. They can remember the steps in how to complete a long division problem or find the value of *x* in a given equation, but they cannot explain why they are completing the problem in that particular way. There is a clear disconnect between *completing* procedures and *understanding* procedures. But in order for students to make meaning in mathematics, they must understand both the how and why.

To adequately meet the needs of your students, it is important to build on their mathematical understanding of each topic covered throughout the year. At its core, mathematics instruction implies learning skills that must build on one another (Dean and Florian 2001). Teachers must help students build on their prior knowledge and apply it to new concepts to deepen students' mathematical understanding. This process can be accomplished by following the steps in the outline below. As this chapter progresses, the steps will be elaborated upon, and specific strategies will be provided for steps 3 and 4.

1. Use mathematical content standards, not your textbook, as a guide for instruction.
2. Find out your students' prior knowledge.
3. Build understanding for each new procedure being taught.
4. Practice each new procedure taught.
5. Assess students to gauge understanding of each new procedure.

Missing Addends



Standard

- solves simple open sentences involving operations on whole numbers

Grades K–2 Lesson

Materials

- counters
- paper and markers or small dry erase boards and dry erase markers
- *Missing Numbers* (page 153; page153.pdf)

Strategies Used in This Lesson

- Think Aloud
- Concrete Materials
- Summarizing
- Multiple exposures

Procedure

1. Display the equation 3 + 5 = 8. Review the other equations in the fact family that can be generated from this number sentence. *(5 + 3 = 8; 8 – 3 = 5; 8 – 5 = 3)*
2. Distribute paper and markers (or dry-erase boards and dry-erase markers) to pairs of students. In pairs, have students complete a fact family for the equation 7 + 2 = 9. When finished, have them display their answers above their heads and discuss them as a class.
3. Based on these fact family relations, lead a discussion with students about how addition and subtraction are related.
4. Distribute 20 counters to each student. Display the number sentence 8 + $\square$ = 12. Tell students that sometimes a number is missing from equations, and fact family relationships can be used to find the missing number. The number is called a *missing addend*.
5. Have students place 12 counters in front of them. Draw 12 counters on the board or display them on the overhead. Then read the think aloud below to model finding the missing addend.

   Think Aloud: To find the missing number in this equation, I can use fact families. My original equation is 8 + $\square$ = 12. I also know that $\square$ + 8 = 12. (As you discuss the equations, write them one below the other on the board or overhead.) The other two equations in the fact family are 12 – $\square$ = 8 and 12 – 8 = $\square$. Now I need to find the equation that can best help me solve the problem. The equation that helps me find the missing number is 12 – 8 = $\square$ because the box is by itself. Now I can use counters to complete the problem. I put the 12 counters out, take 8 counters away, and then count what is left over. (Count the counters aloud.) I have 4 counters left, so the missing number is 4. I can also check this by using counters in the original equation. First I put out 8 counters. Then I put out 4 counters. When I add them together, it equals 12. I found the correct missing number.

Missing Addends *(cont.)*

Procedure *(cont.)*

6. Have students work through the problems below with you. Model the procedures as you work through each equation.
   - 9 + $\square$ = 17
   - 6 + $\square$ = 14
   - 4 + $\square$ = 16
   - $\square$ + 5 = 18
7. Once those problems are finished, have students work with a partner to write and/or draw a summary of the procedure for finding missing addends. Discuss the procedure as a group.
8. In pairs, have students create a problem of their own. Then, in groups of four, have students switch problems and solve them, using their procedure summaries. Once finished, students should share their problems and revise their summaries if necessary.
9. Distribute copies of *Missing Numbers* (page 153) to the students and have them complete it according to the differentiation suggestions below.

Differentiation

Above-Level Learners

Have students work independently to complete problems 5–8 and the challenge problem. In pairs, have students share their alternate algorithms for finding the missing number.

Below-Level Learners

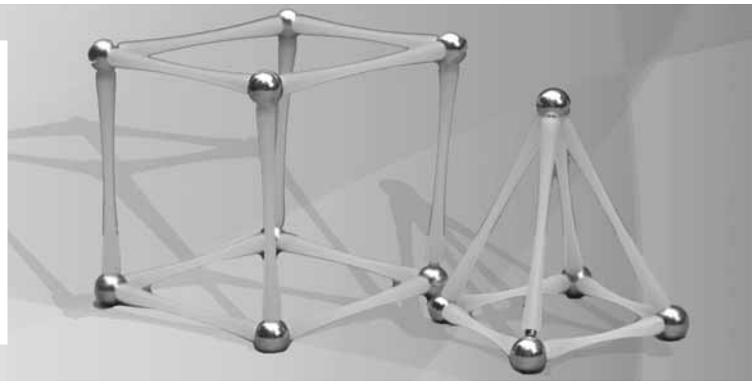
Have students work in pairs to solve problems 1–6. Allow students to use the counters to help them check their work. In groups of four, allow students to work on the challenge problem.

English Language Learners

Have students work in groups of three to complete problems 1–6, using counters to check their work. Bring the students together to work on the challenge problem. Scaffold students' vocabulary and provide them with leading questions as they work on the challenge problem.

Problem-Solving Overview

Teachers must be prepared to approach mathematics instruction through questions, exploration, and problem solving.



According to research, two of the most important skills that students need in order to be prepared for 21st-century careers and citizenship are critical thinking and problem solving. Students need to be prepared to apply their knowledge and seek the right information in order to solve problems. "At the heart of critical thinking and problem solving is the ability to ask the right questions" (Wagner 2008). However, this ability is not inherent. Students must be taught how to question, and they must learn strategies that will help them solve problems, leading to more questions, more problems, and more solutions. And although the ability to vocalize their questions may not be innate, children are innately curious. This curiosity must be channeled and molded so that students can approach and solve problems in creative and meaningful ways.

According to the National Council of Teachers of Mathematics (NCTM 2000), students should be able to:

- build new mathematical knowledge through problem solving.
- solve problems that arise in mathematics and in other contexts.
- apply and adapt a variety of appropriate strategies to solve problems.
- monitor and reflect on the process of mathematical problem solving.

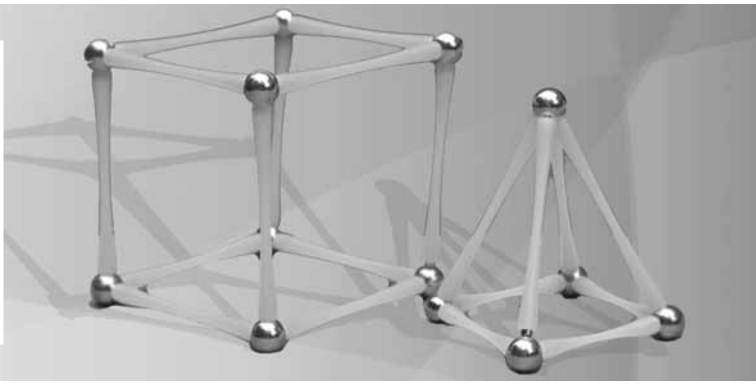
Teachers must be prepared to approach mathematics instruction through questions, exploration, and problem solving. In other words, problem solving is a way of teaching rather than the presentation of word problems. Exposing students to only traditional word problems is not sufficient. By doing so, they are given an unrealistic message about the way mathematics will serve them in the adult world. Most of the problems adults face require mathematical reasoning and skills that are not merely solved by translating the information given into mathematical sentences and then performing the necessary operations.

In order to function in a complex and changing society, it is necessary to be able to solve a wide variety of problems (Wagner 2008). In the real world, problems come in various shapes and forms, many of which involve mathematical concepts and applications. Often, there are numerous possible methods or strategies available to solve the problem. Students need to utilize all the resources they have developed, such as their knowledge, previous experience, and intuition. They then need to analyze, predict, make decisions, and evaluate the outcome of their solutions. For these reasons, it is extremely important that students have mathematics instruction that prepares them to become effective problem solvers.

Drawing a Diagram

Standard

- uses a variety of strategies in the problem-solving process



Background Information

Drawing a diagram is a visual way of processing the information in a problem. It allows students to see what is happening and relate to the situation more clearly.

This strategy often reveals aspects of a problem that may not be apparent at first. If the steps or situations being described in a problem are difficult to visualize, using a diagram may enable the students to see the information more clearly.

There are many types of diagrams that can be used with this strategy. Some of those diagrams include drawing a picture, using symbols or lines to represent objects, and using a time/distance line.

Procedure

Once it is decided that drawing a diagram is the best strategy to use for solving the problem, follow these steps to implement this strategy:

1. Reread the problem.
2. Decide what type of diagram will best show the information in the problem.
3. Draw the diagram according to the scenario in the problem.
4. Check your diagram and solution.
5. Record the solution.

Samples

The following skills and concepts illustrate how drawing a diagram can be used with many different types of problems. Students should be comfortable with them in order to be able to use this problem-solving strategy effectively. Read the sample problem in your grade-level range to see how this strategy can best be implemented in your classroom.

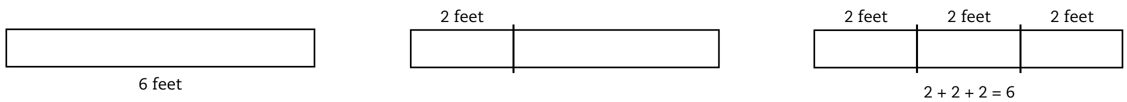
Drawing a Diagram *(cont.)*

Drawing a Picture

Drawing a picture can help students visualize the problem clearly and organize their thoughts.

Grades K–2 Sample

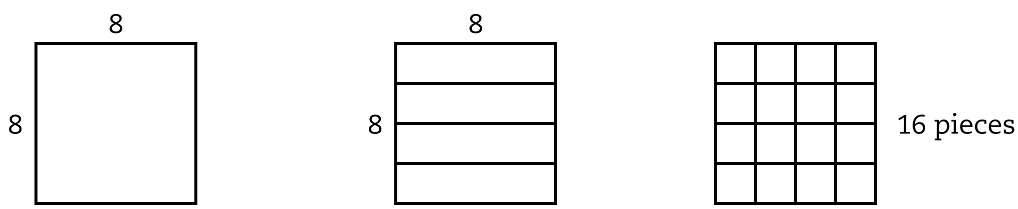
Eva has a ribbon. It is 6 feet long. She wants to cut the ribbon into pieces. She wants each piece to be 2 feet long. How many pieces of ribbon can she cut?



(step 1) (step 2) (step 3)

Grades 3–5 Sample

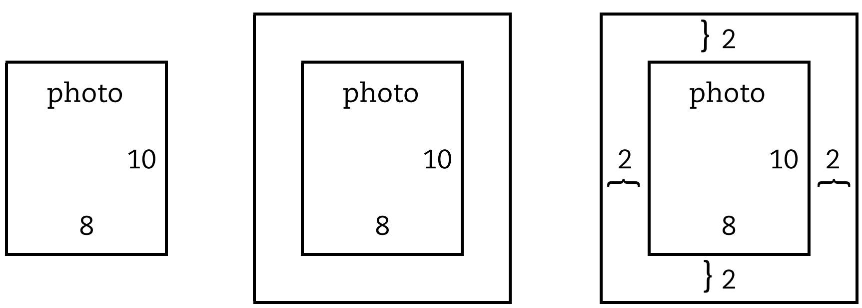
Miguel baked a cake for his mom's birthday. The rectangular pan he used measured 8" x 8". If he cuts 2-inch pieces, how many pieces can he cut?



(step 1) (step 2) (step 3)

Secondary Sample

Shen wants to place a border around a picture of his dad and himself. The picture is 8" wide and 10" tall. The border needs to be 2 inches wider and taller than the picture. What are the dimensions of the border?



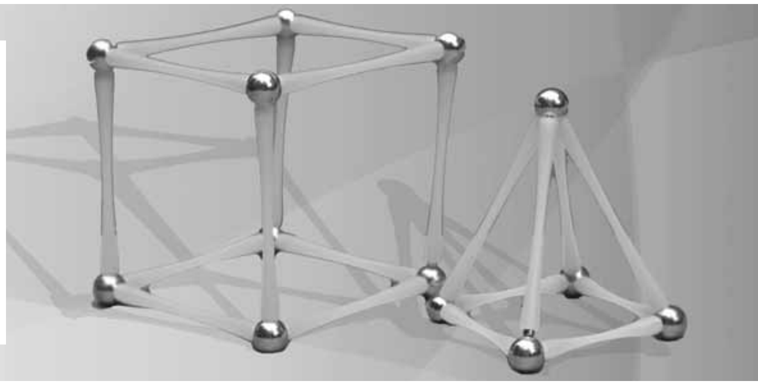
__width__
2 + 8 + 2 = 12 inches

__height__
2 + 10 + 2 = 14 inches

(step 1) (step 2) (step 3) (step 4)

Games Overview

Games help provide a medium for practice that is positive, exciting, and can generate a desire and purpose for learning.



Motivation and Mathematical Learning

Motivation is a student's willingness to give attention, time, energy, and perseverance to learning. It is the willingness to accept the challenge to understand a concept or solve a problem. Motivation is categorized as extrinsic or intrinsic. Extrinsic motivators are things such as treats, grades, and rewards. Intrinsic motivation is the desire for learning and knowledge for its own sake. As teachers, we should take both types of motivation into consideration when planning activities for children.

From an early age, most students are more extrinsically motivated than intrinsically motivated. They work for rewards. However, teachers can increase intrinsic motivation. Students are motivated when:

- they recognize concepts they have learned
- mathematical conversations relate to real-world problems
- tasks and assignments are meaningful and they feel successful

Attitude is also an important part of motivation. Students are motivated to learn when they feel good about a subject and their ability to do well in the subject. There is also a positive correlation between attitude and achievement in mathematics. When students are motivated, they attend to instruction, strive for meaning, and persevere when difficulties arise (Cathcart et al. 2000).

Teachers' attitudes toward mathematics are also influential in forming students' attitudes. The teacher needs to be positive and show enthusiasm for, and interest in, mathematics. (Cathcart et al. 2000) When teachers are positive and enthusiastic, the students are more likely to get on board and reflect the same attitudes.

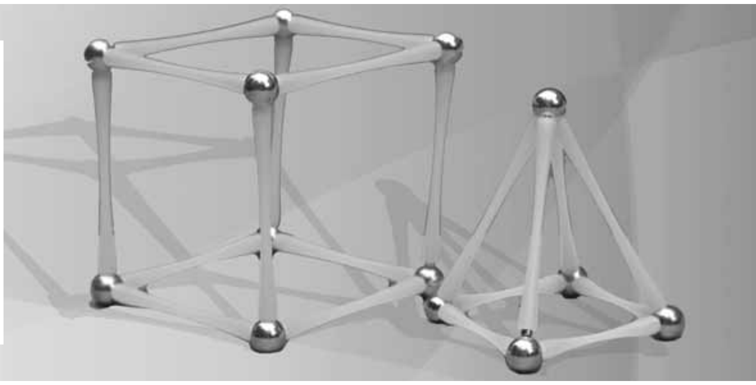
Motivation with Games

Games are a good source of motivation. They are a fun way for students to develop, maintain, and reinforce mastery of basic facts. Games eliminate the tedium of most mathematical drills. They can be used in whole-group, small-group, and individual settings.

In classrooms where competitive games may pose a problem, rules can always be modified so that harmony may rule. However, fair and friendly competition can generate many positive outcomes such as challenge, independence, excitement, and determination. Modeling good sportsmanship is also important. Students need to be shown the proper way to react when they win or lose. They need to be explicitly shown how boasting, teasing, blaming, and anger are never appropriate responses no matter whether one wins or loses.

Assessment Overview

When students understand the expectations and criteria for success they realize that evaluation is part of the learning process and do not feel threatened by it.



Assessment is the means by which we determine what students know and can do. It tells teachers, students, parents, and policymakers something about what students have learned: the mathematical terms they recognize and can use, the procedures they can carry out, the kind of mathematical thinking they do, the concepts they understand, and the problems they can formulate and solve (Kilpatrick, Swafford, and Findell 2001).

It is clear that assessment is a necessary part of education today. All of education is held to high standards, and yearly achievement improvements are of the utmost importance. Assessment for learning provides feedback to improve teaching and learning in the classroom. Although both assessment opportunities have merit, many teachers focus on measuring what students have learned at the end of a unit of instruction and fail to allow opportunities for feedback and revision during instruction (Carr 2008).

In general, there are two types of assessment: summative and formative. According to the Association for Supervision and Curriculum Development (1996), summative assessment is defined as "a culminating assessment, which gives information on students' mastery of content" and formative assessment is defined as "assessment which provides feedback to the teacher for the purpose of improving instruction."

As previously mentioned, summative assessments are usually formal assessments that are given at the end of a unit of study or as a culminating test for the year. These assessments are often paper-and-pencil type tests with multiple choice, matching, free response, essay, or open-ended questions. They are specifically graded, and although they can be used to drive future instruction, they are often used for the purpose of evaluating a student's overall progress.

Formative assessments can be formal or informal. They are primarily used to provide the teacher with feedback on student understanding and progress during a unit of study. Based on the results, the teacher decides if instruction needs to be slowed or quickened to best meet the students' needs. This type of assessment should not be viewed as extra work. "Formative assessment is part of good teaching" and should provide a seamlessness between instruction and assessment (McIntosh 1997).